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The Application of the Method of Quasi-reversibility to the Sideways Heat Equation*

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1. INTRODUCTION

By the *sideways heat equation*, we mean the Cauchy problem for the heat equation in which the temperature and heat flux are specified as function of time at $x = 0$. As we know, the sideways heat equation is an ill-posed problem in the sense of Hadamard. Usual numerical methods, e.g., the finite difference method, would fail to give any reliable results if one tries to compute the solution directly for this problem.

Many investigations have been made into the sideways heat equation. Cannon and Douglas [2] established Hölder continuous dependence on the Cauchy data for solutions of the heat equation that satisfy an *a priori* bound. Payne [8] obtained Hölder continuous dependence on the data for a generalized class of parabolic equation, using a modified weighted energy method. Dorroh [4] studied the nonnegative solutions of the problem and concluded that the admissible Cauchy data are closed under uniform convergence of the data and one first-order derivative of the data. Ginsberg [5] gives a method of constructing the approximate solution of the same problem in L^2 norm. This method truncates the formal solution of Fourier series in the time variable t and Lagrange interpolates the boundary function of the solution by given discrete data at $x = 0$. This approach is proved to be valid by the imposition of an *a priori* bound on the solution. Ames discussed a series of works on the sideways problem for the heat equation in [1].

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The method of quasi-reversibility, which was first proposed by Lattès and Lions in [7], is used to deal with some ill-posed problems. Lattès and Lions did some work on the solution of parabolic equations in Chapters 1 and 5 in [7]. The main idea of this method is that by perturbing the equation in the ill-posed problems, one may obtain a well-posed problem. Then using the information from the solution of the constructed well-posed problem, one solves the original equation, in which the original equation and the new information form another well-posed problem again, and this solution sometimes can be taken to be the approximate solution of the ill-posed problem. Payne [9] introduced this method to yield an "approximate" solution of the backward problem for the heat equation.

In this paper, we will apply the method of quasi-reversibility to the sideways heat equation. In Section 2, a theoretical algorithm is proposed; in Section 3, the algorithm explained in the previous section is proved to be valid under certain assumptions; in Section 4, a numerical method for the sideways heat equation is implemented and some examples are given there. Actually, a direct numerical implementation of the algorithm in Section 2 does not yield good results. Section 4 contains a numerical method that is a modification of the algorithm in Section 2.

The method of quasi-reversibility produces a true solution of the heat equation that satisfies the initial data approximately. Obviously, there are many ways of doing this, so the solution produced by quasi-reversibility is significant only because it also satisfies an *a priori* bound and thus results such as those in [2] and [8] apply.

2. THE METHOD OF QUASI-REVERSIBILITY

Considering the sideways Cauchy problem for the heat equation,

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), && 0 < t < 2\pi, \quad 0 < x < \infty, \\ u(0,t) &= f(t), && 0 < t < 2\pi, \\ u_x(0,t) &= g(t), && 0 < t < 2\pi, \end{aligned} \tag{2.1}$$

we suppose that $f, g \in L^2[0, 2\pi]$, so f and g can be written in their Fourier series, respectively; i.e.,

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{int}, \quad g(t) = \sum_{n=-\infty}^{+\infty} D_n e^{int}, \tag{2.2}$$

with

$$\sum_{n=-\infty}^{\infty} C_n^2 < \infty; \qquad \sum_{n=-\infty}^{\infty} D_n^2 < \infty. \tag{2.3}$$

Then, the sideways problem has the following formal solution:

$$u(x,t) = \sum_{n=-\infty}^{+\infty} \left[C_n e^{int} \cosh\left(\sqrt{in}\, x\right) + \frac{D_n}{\sqrt{in}} e^{int} \sinh\left(\sqrt{in}\, x\right) \right]. \tag{2.4}$$

But in most cases, the above series does not converge for any positive x, since

$$\sqrt{in} = \begin{cases} \sqrt{\dfrac{n}{2}}\,(1+i) & \text{when } n \geq 0, \\ \sqrt{\dfrac{n}{2}}\,(1-i) & \text{when } n \leq 0, \end{cases}$$

and

$$e^{int}\cosh\left(\sqrt{in}\, x\right) \sim O\left(e^{\sqrt{n/2}}\right), \qquad \text{as } n \to \infty$$
$$e^{int}\sinh\left(\sqrt{in}\, x\right) \sim O\left(e^{\sqrt{n/2}}\right), \qquad \text{as } n \to \infty.$$

Now we introduce the method of quasi-reversibility to construct an approximate solution for this problem.

Let $v^{\varepsilon}(x,t)$ be the solution of the following perturbed problem:

$$\begin{aligned} v_{xx}^{\varepsilon}(x,t) &= v_t^{\varepsilon}(x,t) + \varepsilon^2 v_{tt}^{\varepsilon}(x,t), && 0 < t < 2\pi, \quad -\infty < x < \infty, \\ v^{\varepsilon}(0,t) &= f(t), && 0 < t < 2\pi, \\ v_x^{\varepsilon}(0,t) &= g(t), && 0 < t < 2\pi, \end{aligned} \tag{2.5}$$

for small $\varepsilon > 0$. Then

$$\begin{aligned} v^{\varepsilon}(x,t) = \sum_{n=-\infty}^{+\infty} \Bigg[& C_n e^{int} \cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right) \\ & + \frac{D_n}{\sqrt{in - n^2\varepsilon^2}} e^{int} \sinh\left(\sqrt{in - n^2\varepsilon^2}\, x\right) \Bigg], \end{aligned} \tag{2.6}$$

where

$$\sqrt{in - n^2\varepsilon^2} = \begin{cases} \alpha_n + i\beta_n & \text{when } n \geq 0, \\ \alpha_n - i\beta_n & \text{when } n \leq 0, \end{cases} \tag{2.7}$$

with

$$\alpha_n = \frac{1}{\sqrt{2}}\sqrt{\sqrt{n^2 + n^4\varepsilon^4} - n^2\varepsilon^2}, \quad \beta_n = \frac{1}{\sqrt{2}}\sqrt{\sqrt{n^2 + n^4\varepsilon^4} + n^2\varepsilon^2}. \tag{2.8}$$

The real part α_n has a limit of $1/(2\varepsilon)$, as $n \to \infty$. Thus, $v^\varepsilon(x, \cdot)$ is a valid $L^2[0, 2\pi]$ solution for any positive x value. This is the first step of the method.

Then, we will use $v^\varepsilon(x, 0)$ to solve the heat equation on the upper half-plane:

$$\begin{aligned} u_t^\varepsilon(x,t) &= u_{xx}^\varepsilon(x,t), \qquad -\infty < x < \infty, \quad t > 0, \\ u^\varepsilon(x,0) &= v^\varepsilon(x,0), \qquad -\infty < x < \infty, \end{aligned} \tag{2.9}$$

where

$$v^\varepsilon(x,0) = \sum_{n=-\infty}^{+\infty}\left[C_n \cosh\left(\sqrt{in - n^2\varepsilon^2}\,x\right) + \frac{D_n}{\sqrt{in - n^2\varepsilon^2}}\sinh\left(\sqrt{in - n^2\varepsilon^2}\,x\right)\right].$$

Here, the formal solution of this initial value problem for the heat equation is

$$\begin{aligned} u^\varepsilon(x,t) = \sum_{n=-\infty}^{+\infty}\Bigg[& C_n e^{(in - n^2\varepsilon^2)t}\cosh\left(\sqrt{in - n^2\varepsilon^2}\,x\right) \\ & + \frac{D_n}{\sqrt{in - n^2\varepsilon^2}} e^{(in - n^2\varepsilon^2)t}\sinh\left(\sqrt{in - n^2\varepsilon^2}\,x\right)\Bigg]. \end{aligned} \tag{2.10}$$

Next, we will show that $u^\varepsilon(0, t)$ and $u_x^\varepsilon(0, t)$ are close to the given data $f(t)$ and $g(t)$, respectively, which is the main concern about the validity of the method of quasi-reversibility.

3. THEORETICAL ANALYSIS

For further discussion in this paper, we assume that the Fourier coefficients of f satisfy

$$\sum_{n=-\infty}^{\infty} |C_n| < \infty. \tag{3.1}$$

One must note that this is not enough to guarantee that (2.1) has a solution.

To satisfy (3.1), it is sufficient that $f(t)$ be absolutely continuous and have a derivative in L^2. Suppose that the derivative function $f'(t)$ has the Fourier expansion $f'(t) = \sum_{-\infty}^{\infty} C'_n e^{int}$; then $f(t) = \sum_{-\infty}^{\infty} (C'_n/in)e^{int}$. Comparing this Fourier expansion of $f(t)$ with its expansion in (2.2), we may obtain that, for $n = \pm 1, \pm 2, \ldots,$

$$C_n = \frac{C'_n}{in}.$$

Then,

$$\sum_{n=-\infty}^{+\infty} |C_n| = \sum_{n=-\infty}^{\infty} \left| \frac{C'_n}{in} \right| \le \sqrt{\sum_{n=-\infty}^{+\infty} (C'_n)^2} \sqrt{\sum_{n=-\infty}^{\infty} \frac{1}{n^2}} .$$

Claim 1. Under the assumption (3.1), for fixed $x > 0$, and $\varepsilon > 0$, the series in (2.10) is uniformly convergent for t in $[0, 2\pi]$.

Proof. It is enough to obtain the result, if the remainder of the series of (2.10) tends to zero, for fixed $x > 0$, and $\varepsilon > 0$, uniformly in t.

For arbitrary $\eta > 0$, there exists an $N > 0$ such that $\sum_{|n|>N} |C_n| < \eta$. Also by (2.3), we have

$$\begin{aligned}
\sum_{|n|>N} \left| \frac{D_n}{\sqrt{in - n^2\varepsilon^2}} \right| &= \sum_{|n|>N} \frac{|D_n|}{(n^2 + \varepsilon^4 n^4)^{1/4}} \\
&\le \sum_{|n|>N} \frac{|D_n|}{\varepsilon n} \\
&\le \frac{1}{\varepsilon} \left(\sum_{|n|>N} |D_n|^2 \right)^{1/2} \left(\sum_{|n|>N} \frac{1}{n^2} \right)^{1/2} \\
&< \eta,
\end{aligned}$$

for sufficiently large N.

On the other hand, for any integer n, from (2.7), $\cosh(\sqrt{in - \varepsilon^2 n^2}\,x)$ and $\sinh(\sqrt{in - \varepsilon^2 n^2}\,x)$ are bounded by $e^{x/2\varepsilon}$; thus,

$$\sum_{|n|>N}\left[\left|C_n e^{(in-n^2\varepsilon^2)t}\cosh\left(\sqrt{in-n^2\varepsilon^2}\,x\right)\right|\right.$$
$$\left.+\left|\frac{D_n}{\sqrt{in-n^2\varepsilon^2}}e^{(in-n^2\varepsilon^2)t}\sinh\left(\sqrt{in-n^2\varepsilon^2}\,x\right)\right|\right] \le 2\eta e^{x/2\varepsilon},$$

which gives the proof of the claim. ∎

To investigate the distance between $u^\varepsilon(0,t)$ and $f(t)$, one looks at the difference of $u^\varepsilon(0,t) - f(t)$; here is an estimate of it.

Claim 2. Under the assumption (3.1), the following holds:

$$\lim_{\varepsilon\to 0}\left|u^\varepsilon(0,t) - f(t)\right| = 0.$$

Proof. From (2.10),

$$u^\varepsilon(0,t) = \sum_{n=-\infty}^{\infty} C_n e^{int-\varepsilon^2 n^2 t};$$

then,

$$\left|u^\varepsilon(0,t) - f(t)\right| \le \sum_{n=-\infty}^{\infty} |C_n||1 - e^{-n^2\varepsilon^2 t}|.$$

For arbitrary $\eta > 0$, there is an $N > 0$ such that $\sum_{|n|>N}|C_n| < \eta$; thus,

$$\begin{aligned}\left|u^\varepsilon(0,t) - f(t)\right| &= \sum_{|n|>N}|C_n||1-e^{-\varepsilon^2n^2t}| + \sum_{|n|>N}|C_n||1-e^{-\varepsilon^2n^2t}|\\ &\le \sum_{|n|\le N}|C_n||1-e^{-\varepsilon^2n^2t}| + \sum_{|n|>N}|C_n|\\ &\le \left(\sum_{n=-\infty}^{\infty}|C_n|\right)\eta + \eta,\end{aligned}$$

since for each n between $-N$ and N, $1 - e^{-\varepsilon^2 n^2 t}$ converges to 0, uniformly for $t \in [0, 2\pi]$, as ε tends to 0. Thus we prove this claim. ∎

By a similar argument, we also have

Claim 3. If

$$\sum_{n=-\infty}^{+\infty} |D_n| < \infty, \tag{3.2}$$

then

$$\lim_{\varepsilon \to 0} \left| u_x^\varepsilon(0, t) - g(t) \right| = 0.$$

For simplicity, let $g(t) = 0$, i.e., $D_n = 0$ for all integers n in the following theorem.

THEOREM 1. *Assume that* (2.4) *is the solution of* (2.1) *for* $x > 0$, *i.e., the series of the right-hand side of* (2.4) *converges, then both of the series in* (2.10) *and* (2.6) *converge for* $x > 0$ *and* $\varepsilon > 0$; *moreover,*

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = u(x, t), \tag{3.3}$$

$$\lim_{\varepsilon \to 0} v^\varepsilon(x, t) = u(x, t) \tag{3.4}$$

hold for $x > 0$.

Proof. Rewrite the series of $u^\varepsilon(x, t)$ in (2.10) and $v^\varepsilon(x, t)$ in (2.6) as

$$u^\varepsilon(x, t) = \sum_{n=-\infty}^{+\infty} \left[C_n e^{int - n^2\varepsilon^2 t} \cosh\left(\sqrt{in}\, x\right) \frac{\cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right)}{\cosh\left(\sqrt{in}\, x\right)} \right]$$

$$v^\varepsilon(x, t) = \sum_{n=-\infty}^{+\infty} \left[C_n e^{int} \cosh\left(\sqrt{in}\, x\right) \frac{\cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right)}{\cosh\left(\sqrt{in}\, x\right)} \right].$$

To obtain the convergence of $u^\varepsilon(x, t)$ and $v^\varepsilon(x, t)$, we only need to show that for each integer n, $\cosh(\sqrt{in - n^2\varepsilon^2}\, x)/\cosh(\sqrt{in}\, x)$ is uniformly bounded for positive x and ε.

Actually, for arbitrary n, by (2.8),

$$\begin{aligned}
\frac{\left|\cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right)\right|}{\left|\cosh\left(\sqrt{in}\, x\right)\right|} &= \frac{\sqrt{e^{2\alpha_n x} + e^{-2\alpha_n x} + 2\cos 2\beta_n x}}{\sqrt{e^{\sqrt{2n}\, x} + e^{-\sqrt{2n}\, x} + 2\cos\sqrt{2n}\, x}} \\
&= \sqrt{\frac{e^{-(\sqrt{2n} - 2\alpha_n)x} + e^{-(\sqrt{2n} + 2\alpha_n)x} + 2e^{-\sqrt{2n}}\cos 2\beta_n x}{1 + e^{-2\sqrt{2n}\, x} + 2e^{-\sqrt{2n}\, x}\cos\sqrt{2n}\, x}} \\
&\le \frac{2}{\sqrt{1 + 2e^{-\sqrt{2n}\, x}\cos\sqrt{2n}\, x}},
\end{aligned}$$

where α_n and β_n are from (2.8). Note that $\sqrt{2n} - 2\alpha_n > 0$ in the previous estimate and that $2e^{-\sqrt{2n}x}\cos\sqrt{2n}x$ has a minimum value of $-\sqrt{2}e^{-3\pi/4}$, which is a constant less than 1. Thus, we have that $\cosh(\sqrt{in - n^2\varepsilon^2}x)/\cosh(\sqrt{in}x)$ is uniformly bounded for $x, \varepsilon > 0$.

Condition (3.3) holds since for arbitrary small $\eta > 0$, there exists an $N > 0$ that is independent of ε and x such that

$$\left| \sum_{|n|>N} C_n e^{int} \cosh\left(\sqrt{in}\, x\right) \right| < \eta,$$

and

$$\left| \sum_{|n|>N} C_n e^{(in - n^2\varepsilon^2)t} \cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right) \right| < \eta.$$

On the other hand,

$$\left| \sum_{-N}^{N} C_n e^{int} \cosh\left(\sqrt{in}\, x\right) - \sum_{-N}^{N} C_n e^{(in - n^2\varepsilon^2)t} \cosh\left(\sqrt{in - n^2\varepsilon^2}\, x\right) \right| < \eta$$

for sufficiently small ε, since $C_n e^{(in - n^2\varepsilon^2)t}\cosh(\sqrt{in - n^2\varepsilon^2}\,x)$ has a limit of $C_n e^{int}\cosh(\sqrt{in}\,x)$ for each n between $-N$ and N, as $\varepsilon \to 0$. Thus (3.3) is proved. ∎

Similarly, (3.4) can be proved by the same arguments.

Remark. When $g(t) \neq 0$, to have the same result in the above theorem, we need to assume that, in (2.4), $\sum_{n=-\infty}^{+\infty} C_n e^{int}\cosh(\sqrt{in}\,x)$ and $\sum_{n=-\infty}^{+\infty}(D_n/\sqrt{in})e^{int}\sinh(\sqrt{in}\,x)$ should converge.

4. NUMERICAL IMPLEMENTATION

The general idea of the numerical implementation will follow the theoretical method in the previous section with some modification, which is to solve (2.5) numerically by a finite difference method, then using the data of the solution of (2.5) at the line of $t = t_0$ for certain $t_0 > 0$ and $x = \pm x_0$ for certain $x_0 > 0$, one can solve the initial-boundary value problem of the heat equation, which is a well-posed problem.

For the convenience of implementation, one needs the following assumptions:

- In problem (2.5), letting $g(t) = 0$ makes the solution of (2.5) even in x.
- Instead of the time interval $[0, 2\pi]$ in (2.1) or (2.5), the time interval $[0, 1]$ will ease the numerical computation.

Now considering the perturbed problem (2.5) for $x > 0$, we let h and k be the mesh steps for x and t variables, respectively, and let (x_i, t_j) be the mesh points; i.e., $x_i = (i-1)h$, $t_j = (j-1)k$. Since the time interval being considered is $[0,1]$, then, the number of mesh points in time is $m+1$, where m satisfies $k = 1/m$, and the number of mesh points in space is $2n+1$, where n satisfies $h = \bar{x}_0/n$, and $\bar{x}_0$ is the largest x value for all of the mesh points.

Let $v^{\varepsilon}_{i,j} = v^{\varepsilon}(x_i, t_j)$ represent the value of the numerical solution of (2.5) at the mesh point (x_i, t_j). Then the perturbed equation in (2.5) is discretized as

$$\frac{1}{h^2}\delta_x^2 v^{\varepsilon}_{i,j} = \frac{\varepsilon^2}{k^2}\left(\frac{1}{4}\delta_t^2 v^{\varepsilon}_{i-1,j} + \frac{1}{2}\delta_t^2 v^{\varepsilon}_{i,j} + \frac{1}{4}\delta_t^2 v^{\varepsilon}_{i+1,j}\right) + \frac{1}{2k}\tilde{\delta}_t v^{\varepsilon}_{i,j} \quad (4.1)$$

for $i = 2, 3, \ldots, n+1$, and $j = i, i+1, \ldots, m+2-i$, where δ_x and δ_t denote the central difference formula in half mesh step for the partial derivative $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$, respectively; i.e., $\delta_x v^{\varepsilon}_{i,j} = v^{\varepsilon}_{i+1/2,j} - v^{\varepsilon}_{i-1/2,j}$ and $\delta_t v^{\varepsilon}_{i,j} = v^{\varepsilon}_{i,j+1/2} - v^{\varepsilon}_{i,j-1/2}$. Thus, $\delta_x^2 v^{\varepsilon}_{i,j} = v^{\varepsilon}_{i+1,j} - 2v^{\varepsilon}_{i,j} - v^{\varepsilon}_{i-1,j}$ and $\delta_t^2 v^{\varepsilon}_{i,j} = v^{\varepsilon}_{i,j+1} - 2v^{\varepsilon}_{i,j} + v^{\varepsilon}_{i,j-1}$. Finally, $\tilde{\delta}_t$ denotes the central difference formula in the variable t; i.e., $\tilde{\delta}_t v^{\varepsilon}_{i,j} = v^{\varepsilon}_{i,j+1} - v^{\varepsilon}_{i,j-1}$. Actually, (4.1) is not used for the extreme values of j.

In (4.1), we use $(1/h^2)\delta_x^2 v^{\varepsilon}_{i,j}$ to approximate the second derivative term in x, $v^{\varepsilon}_{xx}(x,t)$ at (x_i, t_j); the weighted average of the approximations of the second derivative in t at the mesh points $(x_{i-1}, t_j), (x_i, t_j), (x_{i+1}, t_j)$ to approximate the value of the second derivative of $v^{\varepsilon}(x,t)$ at the point (x_i, t_j); and the central difference approximation to the first derivative in t of $v^{\varepsilon}(x,t)$ at the point (x_i, t_j).

The formula (4.1) yields the following implicit scheme:

$$\begin{aligned} &-\frac{\varepsilon^2 h^2}{4k^2} v^{\varepsilon}_{i+1,j-1} + \left(1 + \frac{\varepsilon^2 h^2}{2k^2}\right) v^{\varepsilon}_{i+1,j} - \frac{\varepsilon^2 h^2}{4k^2} v^{\varepsilon}_{i+1,j+1} \\ &\quad = \left(\frac{\varepsilon^2 h^2}{2k^2} - \frac{h^2}{2k}\right) v^{\varepsilon}_{i,j-1} + \left(2 - \frac{\varepsilon^2 h^2}{k^2}\right) v^{\varepsilon}_{i,j} + \left(\frac{\varepsilon^2 h^2}{2k^2} + \frac{h^2}{2k}\right) v^{\varepsilon}_{i,j+1} \\ &\qquad + \frac{\varepsilon^2 h^2}{4k^2} v^{\varepsilon}_{i-1,j-1} + \left(-1 - \frac{\varepsilon^2 h^2}{2k^2}\right) v^{\varepsilon}_{i-1,j} + \frac{\varepsilon^2 h^2}{4k^2} v^{\varepsilon}_{i-1,j+1}. \end{aligned} \quad (4.2)$$

The mesh points used in (4.2) are shown in Figure 1, where the values of the $\times$ points are used to compute the value of the $\bullet$ points in this implicit scheme.

From Figure 1 we can see that the column of the data of $v^{\varepsilon}_{i+1,\cdot}$ is determined by the previous two columns, i.e., $v^{\varepsilon}_{i,\cdot}$ and $v^{\varepsilon}_{i-1,\cdot}$, which means

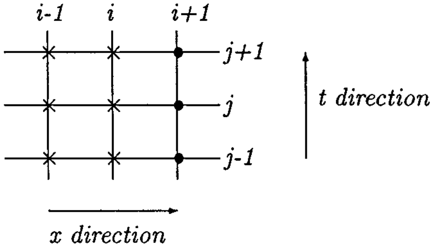

FIG. 1. Implicit scheme for (2.5).

that we need the first two columns of data, i.e., the data at $x = 0$ $(i = 1)$ and $x = h$ $(i = 2)$, to do the computation for the solution.

By the given conditions $v^{\varepsilon}(0, t) = f(t)$ and $v_x^{\varepsilon}(0, t) = g(t) = 0$, we have $v_{1,j}^{\varepsilon}$ from $f(t)$. The data for $v_{2,j}^{\varepsilon}$ are approximated by the Taylor expansion of $v^{\varepsilon}(x, t)$ at $x = h$:

$$v^{\varepsilon}(h, t) = v^{\varepsilon}(0, t) + h \cdot v_x^{\varepsilon}(0, t) + \frac{h^2}{2} \cdot v_{xx}^{\varepsilon}(0, t) + O(h^3)$$

or

$$v^{\varepsilon}(h, t) = v^{\varepsilon}(0, t) + \frac{h^2}{2} \cdot \left(\varepsilon^2 v_{tt}^{\varepsilon}(0, t) + v_t^{\varepsilon}(0, t)\right) + O(h^3), \quad (4.3)$$

since $v_x^{\varepsilon}(0, t) = g(t) = 0$ and by the equation (2.5). The discretization of (4.3) is

$$v_{2,j}^{\varepsilon} = v_{1,j}^{\varepsilon} + \frac{\varepsilon^2 h^2}{2k^2} \cdot \left(v_{1,j+1}^{\varepsilon} - 2v_{1,j}^{\varepsilon} + v_{1,j-1}^{\varepsilon}\right) + \frac{h^2}{4k}\left(v_{1,j+1}^{\varepsilon} - v_{1,j-1}^{\varepsilon}\right) \quad (4.4)$$

for $j = 2, 3, \ldots, m$. Here, we may not get the data for $v_{2,1}^{\varepsilon}$ and $v_{2,m+1}^{\varepsilon}$ by the above approximation. As a matter of fact, we are losing two ending points at $i = 2$ (see Fig. 2).

Now, let us look into the system of equations to solve $v_{3,\cdot}^{\varepsilon}$. When (4.2) is used to solve for $v_{3,\cdot}^{\varepsilon}$, i has a value of 2, and j goes from 4 to $m - 2$; i.e., we have $m - 5$ equations in the system. But the unknown variables involved in the system are $v_{3,3}^{\varepsilon}, v_{3,4}^{\varepsilon}, \ldots,$ and $v_{3,m-1}^{\varepsilon}$, which is $m - 3$ in total, shown in Figure 2.

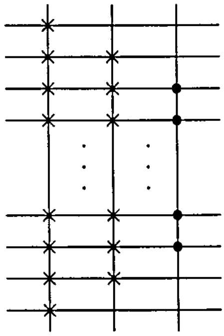

FIG. 2. The mesh points in the first three columns.

To make the system solvable, we will use an explicit scheme at $j = 3$ and $j = m - 1$, respectively, to obtain another two equations. They are

$$v_{3,3}^{\varepsilon} = \left(\frac{\varepsilon^2 h^2}{k^2} - \frac{h^2}{2k} \right) v_{2,2}^{\varepsilon} + \left(2 - \frac{2\varepsilon^2 h^2}{k^2} \right) v_{2,3}^{\varepsilon} + \left(\frac{\varepsilon^2 h^2}{k^2} + \frac{h^2}{2k} \right) v_{2,4}^{\varepsilon} - v_{1,3}^{\varepsilon}$$

and

$$v_{3,m-1}^{\varepsilon} = \left(\frac{\varepsilon^2 h^2}{k^2} - \frac{h^2}{2k} \right) v_{2,m-2}^{\varepsilon} + \left(2 - \frac{2\varepsilon^2 h^2}{k^2} \right) v_{2,m-1}^{\varepsilon} + \left(\frac{\varepsilon^2 h^2}{k^2} + \frac{h^2}{2k} \right) v_{2,m}^{\varepsilon} - v_{1,m-1}^{\varepsilon}.$$

Now the number of the unknown variables matches the number of the equations in the system.

Generally, two mesh points are lost when we compute the data for a new column. Thus, the computing area looks like the diagram in Figure 3. Thus, the numerical computation reflects the well-known "domain of influence" of initial data for the wave equation (see [6]).

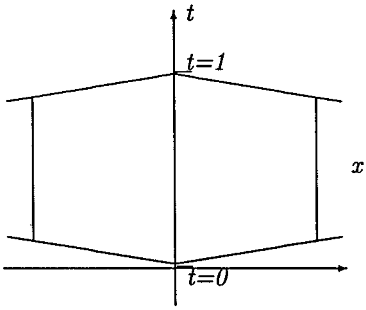

FIG. 3. Computing area for (2.5).

Let $V_i^\varepsilon = \{v_{i,i}^\varepsilon, v_{i,2}^\varepsilon, \ldots, v_{i,m-i+2}^\varepsilon\}^T$, then (4.2) can be rewritten in matrix form:

$$A \cdot V_{i+1}^\varepsilon = B \cdot V_i^\varepsilon + C \cdot V_{i-1}^\varepsilon, \tag{4.5}$$

where

$$A = \begin{pmatrix} 1-\frac{r}{2} & \frac{5r}{4} & -r & \frac{r}{4} & \cdots & 0 & 0 & 0 & 0 \\ -\frac{r}{4} & 1+\frac{r}{2} & -\frac{r}{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\frac{r}{4} & 1+\frac{r}{2} & -\frac{r}{4} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{r}{4} & 1+\frac{r}{2} & -\frac{r}{4} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{r}{4} & 1+\frac{r}{2} & -\frac{r}{4} \\ 0 & 0 & 0 & 0 & \cdots & \frac{r}{4} & -r & \frac{5r}{4} & 1-\frac{r}{2} \end{pmatrix} \tag{4.6}$$

$$B = \begin{pmatrix} \frac{r-s}{2} & 2-r & \frac{r+s}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{r-s}{2} & 2-r & \frac{r+s}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{r-s}{2} & 2-r & \frac{r+s}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{r-s}{2} & 2-r & \frac{r+s}{2} \end{pmatrix} \tag{4.7}$$

$$C = \begin{pmatrix} \frac{r}{4} & -1-\frac{r}{2} & \frac{r}{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{r}{4} & -1-\frac{r}{2} & \frac{r}{4} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{r}{4} & -1-\frac{r}{2} & \frac{r}{4} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{r}{4} & -1-\frac{r}{2} & \frac{r}{4} \end{pmatrix}, \quad (4.8)$$

with $r = \varepsilon^2 h^2/k^2$, and $s = h^2/k$.

Conceptually, (4.5) can be solved in the form

$$V_{i+1}^{\varepsilon} = A^{-1}B \cdot V_i^{\varepsilon} + A^{-1}C \cdot V_{i-1}^{\varepsilon}$$

or

$$V_{i+1}^{\varepsilon} = (A^{-1} \cdot B \quad A^{-1} \cdot C) \cdot \begin{pmatrix} V_i^{\varepsilon} \\ V_{i-1}^{\varepsilon} \end{pmatrix}.$$

It is important here to consider the stability of the above linear equation, which is the stability of the matrix on the right side in the equation,

$$L = (A^{-1} \cdot B \quad A^{-1} \cdot C). \quad (4.9)$$

So far, we are not able to use any theorem to prove that the matrix (4.9) has any kind of norm that is less than 1. But by numerical computing, if we fill in several extra zero rows at the bottom of the matrix (4.9) to make it square,

$$L' = \begin{pmatrix} A^{-1} \cdot B & A^{-1} \cdot C \\ 0 & 0 \end{pmatrix}, \quad (4.10)$$

then the maximum absolute value of all of the eigenvalues of L' will be less than 1. For example, if the size of (4.10) is about 400×400, the eigenvalue of (4.10) is about 0.4–0.5, although all of the norms of both $A^{-1} \cdot B$ and $A^{-1} \cdot C$ are larger than 1.

In this way, we compute the data on the right side of the computing area shown in Figure 3. Using the even property of the solution we mentioned previously, the computational results on the whole computing area are obtained. We will use these data to form a boundary-initial value problem for the heat equation, which is the second step of the computation.

Suppose that the right-side boundary of the computing area in Figure 3 is at $x = \bar{x}_0$. In the second part of the computation, the numerical solution of the following problem is considered:

$$\begin{aligned} u_t^\varepsilon(x,t) &= u_{xx}^\varepsilon(x,t), && t_0 < t < T_0, \quad -x_0 < x < x_0, \\ u^\varepsilon(x,t_0) &= v^\varepsilon(x,t_0), && -x_0 < x < x_0, \\ u^\varepsilon(\pm x_0,t) &= v^\varepsilon(\pm x_0,t), && t_0 < t < T_0, \end{aligned} \tag{4.11}$$

for some $t_0 \geq 0$ and $0 \leq x_0 \leq \bar{x}_0$, where $v^\varepsilon(x,t)$ is given as discretized data, which is computed in the previous step.

Let $u_{i,j}^\varepsilon$ denote the discretized data of $u^\varepsilon(x_i,t_j)$. The heat equation can be discretized as

$$\frac{1}{k}\delta_t u_{i,j+\frac{1}{2}}^\varepsilon = \frac{1}{2h^2}\left(\delta_x^2 u_{i,j}^\varepsilon + \delta_x^2 u_{i,j+1}^\varepsilon\right),$$

which yields the following implicit scheme, called the Crank−Nicholson scheme:

$$\begin{aligned} -\frac{k}{2h^2}u_{i-1,j+1}^\varepsilon &+ \left(1+\frac{k}{h^2}\right)u_{i,j+1}^\varepsilon - \frac{k}{2h^2}u_{i+1,j+1}^\varepsilon \\ &= \frac{k}{2h^2}u_{i-1,j}^\varepsilon + \left(1-\frac{k}{h^2}\right)u_{i,j}^\varepsilon + \frac{k}{2h^2}u_{i+1,j}^\varepsilon \end{aligned} \tag{4.12}$$

This can be proved to be an unconditionally stable scheme by von Neumann's method.

EXAMPLE 1. Let $f(t) = 1 - e^{-t}$, and $g(t) = 0$.

The exact solution $u(x,t)$ of the heat equation with $u(0,t) = f(t)$ and $u_x(0,t) = g(t)$ is $u(x,t) = 1 - e^{-t}\cos x$. In the computation, we will choose $k = 0.01$, $h = 0.01$. Let $x_0 = 1$ and $\varepsilon = 0.01$. Just as in the previous explanation, we solve the damped wave equation (2.5). The area that we could obtain numerical data for (2.5) is the rombus among $(x,t) = (\pm 0.5, 0)$ and $(\pm 0.5, 1)$. As the second step in computing the solution of the initial-boundary value problem for the heat equation, we start at the line of $t = 0.3$ with a boundary of x at $x = \pm 0.3$. In this numerical experiment, a relative random error of $[-0.5 \times 10^{-4}, 0.5 \times 10^{-4}]$ is added to $f(t)$; Table 1 and Figure 4 show the relative error for $0.30 \leq t \leq 0.70$.

An effort was also made to implement the theoretical method more directly by using finite Fourier transforms. However, this did not yield good results.

EXAMPLE 2. Let $f(t) = 1 - \cos t$, and $g(t) = 0$.

The exact solution $u(x,t)$ of the heat equation with $u(0,t) = f(t)$ and $u_x(0,t) = g(t)$ is $u(x,t) = 1 - \frac{1}{2}(e^{x/\sqrt{2}}\cos(x/\sqrt{2} + t) + e^{-x/\sqrt{2}}\cos(x/\sqrt{2} - t))$. By using the same parameter values for h, k, x_0, and ε, and the same range of perturbation on the given exact data of $f(t)$, similar good results are also obtained (Table 2 and Fig. 5).

TABLE 1
Maximum Relative Error for $0.30 \le t \le 0.70$ for Example 1

x	Max relative error
0.00	3.1047e − 05
0.05	3.0965e − 05
0.10	3.0143e − 05
0.15	4.3827e − 05
0.20	6.7706e − 05
0.25	1.2398e − 04
0.30	2.2326e − 04

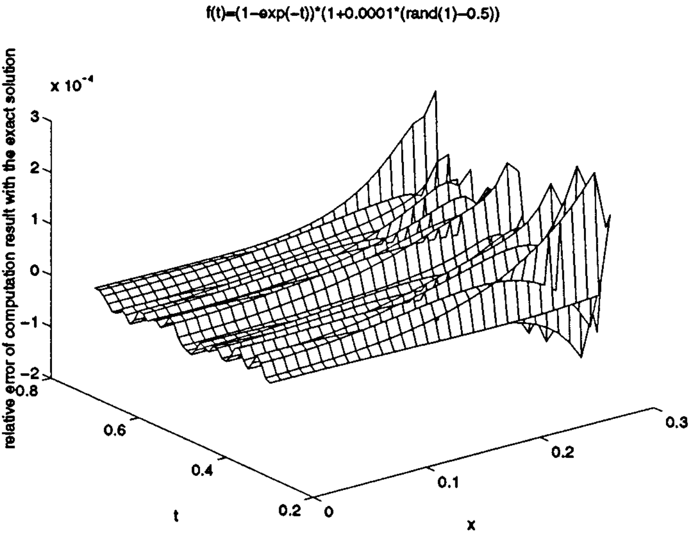

FIG. 4. The relative difference between the exact data and computed data for Example 1.

Cannon and Ewing presented a direct numerical method for the same problem in [3].

5. CONCLUSIONS

In this paper, we use the method of quasi-reversibility to construct an approximate solution of the ill-posed sideways heat equation. We also prove that, by putting certain restrictions on the boundary conditions, the solution obtained from this method converges to the exact solution, if the

TABLE 2
Maximum Error for $0.30 \leq t \leq 0.70$ for Example 2

x	Max relative error
0.00	7.9138e − 05
0.05	8.0244e − 05
0.10	8.1923e − 05
0.15	9.2092e − 05
0.20	1.1708e − 04
0.25	1.8058e − 04
0.30	2.3025e − 04

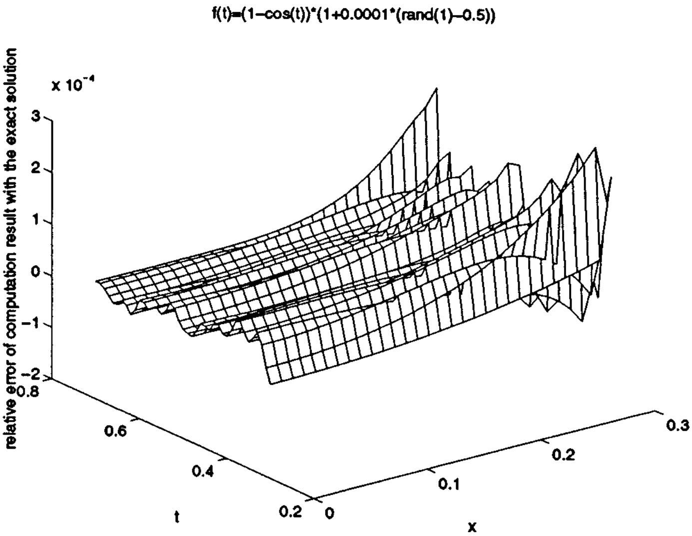

FIG. 5. The relative difference between the exact data and computed data for Example 2.

exact solution of the problem exists. On the other hand, if the boundary conditions of the problem, $f(t)$ and $g(t)$, are not "good enough" to make an exact solution exist, this method could still give a theoretical approximate solution, which gives a small difference from the given data on the boundary.

Moreover, based on the method of quasi-reversibility, we use the finite difference method to compute the numerical solution of the sideways heat equation. The result shows that the numerical solution matches the exact solution quite well in certain (x, t) areas.

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